Math 245B Lecture 24 Notes

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1 Distributions, Weak L^p, Strong Type, and Weak Type

1.1 Distributions

Last time, we introduced the notion of a distribution function $\lambda_f(\alpha) = \mu(\{|f| > \alpha\})$.

Definition 1.1. Let $f \geq 0$ and $\mu(X) < \infty$. Then the **distribution** of f is the measure $\nu(E) = \mu(\{x \in X : f(x) \in E\})$.

Observe that

$$\nu(a, b) = \mu(\{a < f \leq b\}) = \lambda_f(a) - \lambda_f(b) = [-\lambda_f(b)] - [-\lambda_f(a)].$$

So λ_f determines the measure ν and basically contains all the information about how much measure the range of f has in given sets.

Proposition 1.1 (Chebyshev's inequality)**.** *Let $0 < p < \infty$, and let $f \in L^p$. Then $\lambda_f(\alpha) \leq \|f\|_p^p/\alpha^p$.*

Remark 1.1. When $p = 1$, this is called Markov's inequality.[1]

Proof. $\lambda_f(\alpha) = \mu(\{f > \alpha\}) =: \mu(E_\alpha)$. By definition, $\mathbb{1}_{E_\alpha}\alpha^p \leq |f|^p$. Then

$$\mu(E_\alpha) = \alpha^p \int \mathbb{1}_{E_\alpha} \, d\mu \leq \int |f|^p \, d\mu. \qquad \square$$

1.2 Weak L^p

Definition 1.2. If $f : X \to \mathbb{C}$, the **"weak L^p norm** of f is

$$[f]_p = (\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha))^{1/p}.$$

[1]Markov was Chebyshev's advisor. Chebyshev is responsible for noticing that the inequality holds in general.

Remark 1.2. This is generally not actually a norm; the triangle inequality fails. Chebyshev's inequality says that

$$[f]_p \leq \|f\|_p.$$

Definition 1.3. The **weak L^p space** is

$$\operatorname{wk} L^p(\mu) = \{f : X \to \mathbb{C} \mid [f]_p < \infty\}/\sim,$$

under the equivalence relation of μ-a.e. equality.

By Chebyshev's inequality, $\operatorname{wk} L^p \supseteq L^p$.

Example 1.1. Let m be Lebesgue measure on $(0, \infty)$. Consider $f(x) = x^{-1/p}$. Then $f \notin L^p(m)$. But

$$[f_p] = \sup_\alpha m(\{f > \alpha\}) = \sup_\alpha \alpha^p = \sup_\alpha m([0, 1/\alpha^p))\alpha^p = 1$$

Proposition 1.2. *Let $0 < p < \infty$, and let $f : X \to \mathbb{C}$. Then*

$$\|f\|_p^p = \int |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.$$

Proof. If there exists α such that $\lambda_f(\alpha) = \infty$, the the right hand side is infinite. By Chebyshev's inequality, so is the left hand side. So we may assume that $\lambda_f(\alpha) < \infty$ for all α. Then $\{f \neq 0\}$ is σ-finite. So we may assume μ is σ-finite.

Now consider $E = \{(x, y) \in X \times [0, \infty) : y < |f(x)|^p\}$. Now, by Tonelli's theorem,

$$\begin{aligned}
\int_X |f|^p \, d\mu &= \int_X \int_0^{|f(x)|^p} dy \, d\mu(x) \\
&= (\mu \otimes m)(E) \\
&= \int_0^\infty \mu(\{|f|^p > y\}) = p \int \alpha^{p-1} \lambda_f(\alpha) \, d\alpha,
\end{aligned}$$

where we have used the substitution $y = \alpha^p$. $\square$

1.3 Strong type and weak type

Definition 1.4. Let $\mathcal{D}$ be some vector space of measurable $\mathbb{C}$-valued functions on $(X, \mathcal{M}, \mu)$, and let $T : \mathcal{D} \to L^0(Y, \mathcal{N}, \nu)$ (the space of measurable functions. T is **sublinear** if

1. $c > 0 \implies |T(cf)| = c|Tf|$ for all $f \in \mathcal{D}$
2. $|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|$.

Example 1.2. Let $\mathcal{D} = L^1_{\text{loc}}$. The **Hardy-Littlewood maximal operator** is

$$Hf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy.$$

Then $H(f_1 + f_2) \le Hf_1 + Hf_2$, so H is sublinear.

Remark 1.3. Often, sublinear functions arise from taking the pointwise supremum of a collection of linear functions.

Definition 1.5. T is **strong type** (p,q) for $1 \le p, q \le \infty$ if

1. $L^p(\mu) \subseteq \mathcal{D}$
2. $T[L^p(\mu)] \subseteq L^q(\nu)$, and $\|Tf\|_q \le C\|f\|_p$ for some fixed $C > 0$.

Definition 1.6. T is **weak type** (p,q) for $1 \le p, q \le \infty$ if

1. $T[L^p(\mu)] \subseteq \operatorname{wk} L^q(\nu)$
2. $[Tf]_q \le C[f]_p$ for some fixed $C > 0$.

Example 1.3. H is strong type $(, \infty, \infty)$. The Hardy-Littlewood maximal inequality says that H is weak type $(1,1)$.

Theorem 1.1 (special case of Marciukiewicz' theorem)**.** *Suppose T is sublinear on $\mathcal{D} = L^1(\mu) + L^\infty(\mu)$. Suppose T is weak type $(1,1)$ and strong type (∞,∞). Then T is strong type (p,p) for all $p \in (0,\infty]$.*

Remark 1.4. $L^1(\mu) + L^\infty(\mu) \supseteq L^p(\mu)$ for all p.

Example 1.4. The Hardy-Littlewood maximal operator is strong type (p,p) for all $p \in [0,\infty]$. This is very difficult to prove by hand.

Proof. Pick a C such that $\|Tf_\infty \le C\|f_\infty$ and $[Tf]_1 \le C\|f\|_1$ for all $f \in L^\infty$ or L^1. Let $f \in L^p(\mu$, and let $A > 0$. Write $f = f_1 + f_2$, where $f_1 = f\mathbb{1}_{\{|f|>A\}}$ and $f_2 = f\mathbb{1}_{\{|f|\le A\}}$. We will optimize over the value of A. We have

$$\|f_1\| = \int_{\{|f|>A\}} |f| \le \int_{\{|f|>A\}} \frac{|f|^p|}{A^{p-1}} = \frac{A^{p-1}}{\int}_{\{|f|>A\}} |f|^p < \infty.$$

By subliniearity,

$$|Tf(x)| \le |Tf_1(x)| + \underbrace{|Tf_2(x)|}_{\le CA}.$$

So

$$\mu(\{|Tf| > 2CA\}) \le \mu(\{|Tf_1 > CA\}) \le \frac{C\|f_1\|}{CA} = \frac{1}{A}\int_{\{|f|>A\}} |f|.$$

So we have improved the weak type $(1,1)$ inequality to get

$$\lambda_{Tf}(2CA) \leq \frac{1}{A} \int_{\{|f|>A\}} |f|.$$

Substitute this expression into the following:

$$\|Tf\|_p^p = p \int_0^\infty \alpha^{p-1} \lambda_{Tf}(\alpha)\, d\alpha \leq (2C)^p \frac{p}{p-1} \|f\|_p^p. \qquad \square$$